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WIENER'S LEMMA FOR BANACH ALGEBRAS OF INFINITE MATRICES OF GBS CLASS

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Abstract. In this paper, we introduce a Banach algebra of infinite matrices of GBS class, which is inverse-closed in the Banach algebra $\mathcal{B}(\ell^2)$ of all bounded linear operators on ℓ^2. Here infinite matrices are defined on a general index set Λ which may not form a group.

1. Introduction

N. Wiener showed in [23] that if f is a periodic function with an absolutely convergent Fourier series and it vanishes nowhere on the real line, then $1/f$ has an absolutely convergent Fourier series too. This is now called the classical Wiener's lemma.

Define the *GBS(Gohberg-Baskakov-Sjöstrand) class* $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ \big(a(i,j)\big)_{i,j\in\mathbb{Z}^d} : \sum_{k\in\mathbb{Z}^d} \big(\sup_{i-j=k} |a(i,j)|\big) < \infty \Big\}. \tag{1.1}$$

Then the classical Wiener's lemma can be reformulated as follows: $\mathcal{C}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$, the space of all bounded linear operators on the space ℓ^2 of square-summable sequences. Here a Banach algebra $\mathbb{B}$, which is a subalgebra of $\mathbb{A}$, is called *inverse-closed* if any $A \in \mathbb{B}$ with the inverse $A^{-1} \in \mathbb{A}$ implies $A^{-1} \in \mathbb{B}$. In this case the spectrum of $A \in \mathbb{B}$ in the algebra $\mathbb{B}$ is identical to the spectrum of A in the algebra $\mathbb{A}$.

Wiener's lemma has various extensions and applications. For a weight matrix w, Define *Gröchenig-Schur class* $\mathcal{S}_w(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$\begin{aligned}\mathcal{S}_w(\mathbb{Z}^d, \mathbb{Z}^d) :=& \Big\{ \big(a(i,j)\big)_{i,j\in\mathbb{Z}^d} : \\ & \max\Big(\sup_{i\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} |a(i,j)|w(i,j), \sup_{j\in\mathbb{Z}^d} \sum_{i\in\mathbb{Z}^d} |a(i,j)|w(i,j)\Big) < \infty \Big\};\end{aligned} \tag{1.2}$$

and the *Beurling class* $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) := \Big\{ \big(a(i,j)\big)_{i,j\in\mathbb{Z}^d} : \sum_{k\in\mathbb{Z}^d} \big(\sup_{|i-j|\geq|k|} |a(i,j)|\big) < \infty \Big\}, \tag{1.3}$$

where we set $|x| = \max(|x_1|, \cdots, |x_d|)$ for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. The unweighted Gröchenig-Schur class, that is, $w \equiv 1$, is not inverse-closed in $\mathcal{B}(\ell^2)$([22]), but when the weight w satisfies the GRS-condition, the weighted Gröchenig-Schur class

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$\mathcal{S}_w(\mathbb{Z}^d, \mathbb{Z}^d)$ is inverse-closed in $\mathcal{B}(\ell^2)$ ([2, 6, 7, 9, 17, 19, 22]). The GBS class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ and the Beurling class $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ are inverse-closed in $\mathcal{B}(\ell^2)$ ([3, 16, 19, 20]). The inverse-closed property has important applications in dual wavelet frames, dual Gabor frames, algebra of pseudo-differential operators, and reconstruction in sampling theory ([1, 4, 5, 8, 10, 12, 14, 18, 21]). The reader may refer to [6, 11, 15] and references therein for historical remarks, recent advances and applications.

In this paper, we consider infinite matrices defined on a general index set $\Lambda \subset \mathbb{R}^d$ satisfying

$$\alpha = \sup_{k\in\mathbb{Z}^d} \sum_{\lambda\in\Lambda} \chi_{k+[-2,2)^d}(\lambda) < \infty. \tag{1.4}$$

The set Λ may not form a group. Our prime models are paraboloids

$$\{(x, y, z) : \ z = ax^2 + by^2, x, y \in \mathbb{Z}\}$$

and elliptical hyperboloids

$$\{(x, y, z) : \ z^2 = ax^2 + by^2, x, y \in \mathbb{Z}^2\},$$

where $a, b > 0$.

In this paper, we prove that for a weight w satifying (2.4) and (2.16), and $1 \le q \le \infty$, the GBS classes $\mathcal{C}_{q,w}(\Lambda, \Lambda)$ (refer to Defintion 2.1) of infinite matrices defined on a general index set Λ are inverse-closed Banach algebras of $\mathcal{B}(\ell^2)$.

Notation: For $1 \le q \le \infty$, we denote by q' the conjugate exponent of q, that is, $1/q + 1/q' = 1$ and we write the constant weight matrix $u_0 \equiv 1$, that is, every element of u_0 is 1.

2. Wiener's Lemma for Banach alagebra of GBS class

In this section, let $1 \le q \le \infty$ and we consider the inverse-closedness of a Banach algebra $\mathcal{C}_{q,w}(\Lambda, \Lambda)$ in $\mathcal{B}(\ell^2)$.

We say that w is a weight if

$$w(\lambda, \lambda') \ge 1 \qquad \text{for any } \lambda, \lambda' \in \Lambda, \tag{2.1}$$

$$w(\lambda, \lambda') = w(\lambda', \lambda) \qquad \text{for any } \lambda, \lambda' \in \Lambda, \tag{2.2}$$

and

$$\sup_{\lambda\in\Lambda} w(\lambda, \lambda) < \infty. \tag{2.3}$$

For a weight w, the weight u is called *a companion matrix* of w if

$$w(\lambda, \lambda') \le w(\lambda, \tilde{\lambda})u(\tilde{\lambda}, \lambda') + u(\lambda, \tilde{\lambda})w(\tilde{\lambda}, \lambda') \quad \text{for all } \lambda, \lambda', \tilde{\lambda} \in \Lambda. \tag{2.4}$$

Definition 2.1. *Let $1 \le q \le \infty$ and w be a weight matrix. Define the GBS class*

$$\mathcal{C}_{q,w}(\Lambda, \Lambda) = \{A = (a(\lambda, \lambda'))_{\lambda,\lambda'\in\Lambda} : \|A\|_{\mathcal{C}_{q,w}} < \infty\}, \tag{2.5}$$

where for $1 \le q < \infty$,

$$\|A\|_{\mathcal{C}_{q,w}} := \alpha\Big(\sum_{k\in\mathbb{Z}^d} \sup_{\lambda,\lambda'\in\Lambda} |a(\lambda, \lambda')|^q w(\lambda, \lambda')^q \chi_{k+[0,1)^d}(\lambda - \lambda')\Big)^{1/q}, \tag{2.6}$$

for $q = \infty$,

$$\|A\|_{\mathcal{C}_{\infty,w}} = \sup_{\lambda,\lambda'\in\Lambda} |a(\lambda, \lambda')| w(\lambda, \lambda') \tag{2.7}$$

and α is the positive real number in (1.4).

When $q = \infty$, $\mathcal{C}_{\infty,w}$ is called *the Jaffard class.* Briefly we write $\mathcal{C}_{q,w}$ for $\mathcal{C}_{q,w}(\Lambda, \Lambda)$.
From the definition, we can easily see that for $A, B \in \mathcal{C}_{q,w}$, $A + B \in \mathcal{C}_{q,w}$ and $\|A + B\|_{\mathcal{C}_{q,w}} \le \|A\|_{\mathcal{C}_{q,w}} + \|B\|_{\mathcal{C}_{q,w}}$.
Note that for an index set $\Lambda \subset \mathbb{R}^d$ satisfying (1.4),

$$\max\Big\{\sup_{\lambda'\in\Lambda}\sum_{\lambda\in\Lambda}\chi_{k+(-1,1)^d}(\lambda-\lambda'),\ \sup_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda}\chi_{k+(-1,1)^d}(\lambda-\lambda')\Big\} \le \alpha. \tag{2.8}$$

The Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{q,w}$ of infinite matrices has the following basic properties.

Proposition 2.2. *Let $1 \le q \le \infty$ and w be a weight matrix on $\Lambda \times \Lambda$.*

(i) *If $A \in \mathcal{C}_{q,w}$, then $cA \in \mathcal{C}_{q,w}$ for any $c \in \mathbb{R}$, and $\|cA\|_{\mathcal{C}_{q,w}} = |c|\|A\|_{\mathcal{C}_{q,w}}$.*
(ii) *If $A \in \mathcal{C}_{q,w}$ and $\sup_{\lambda\in\Lambda}\|w^{-1}(\lambda,\cdot)\|_{\ell^{q'}} < \infty$, then $\|A\|_{\mathcal{B}(\ell^2)} \le \sup_{\lambda\in\Lambda}\|w^{-1}(\lambda,\cdot)\|_{\ell^{q'}}\|A\|_{\mathcal{C}_{q,u}}$ so $A \in \mathcal{B}(\ell^2)$.*
(iii) *Let u be a weight matrix on $\Lambda \times \Lambda$. If $A \in \mathcal{C}_{q,w}$ and $uw^{-1} \in \mathcal{C}_{q',u_0}$, then*

$$\|A\|_{\mathcal{C}_{1,u}} \le \alpha^{-1}\|uw^{-1}\|_{\mathcal{C}_{q',u_0}}\|A\|_{\mathcal{C}_{q,w}}. \tag{2.9}$$

Proof. (i) Trivial.
(ii)Note that for any $\lambda' \in \Lambda$

$$\begin{aligned}\sum_{\lambda\in\Lambda}|a(\lambda,\lambda')| &\le \big(\sum_{\lambda\in\Lambda}|a(\lambda,\lambda')|^q|w(\lambda,\lambda')|^q\big)^{1/q}\big(\sum_{\lambda\in\Lambda}|w(\lambda,\lambda')|^{-q'}\big)^{1/q'}\\ &\le \|A\|_{\mathcal{C}_{q,w}}\sup_{\lambda'\in\Lambda}\|w^{-1}(\cdot,\lambda')\|_{\ell^{q'}}\end{aligned}$$

and similarly for any $\lambda \in \Lambda$, $\sum_{\lambda'\in\Lambda}|a(\lambda,\lambda')| \le \|A\|_{\mathcal{C}_{q,w}}\sup_{\lambda\in\Lambda}\|w^{-1}(\lambda,\cdot)\|_{\ell^{q'}}$.
Since

$$\|A\|_{\mathcal{B}(\ell^2)} \le \max(\sup_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda}|a(\lambda,\lambda')|, \sup_{\lambda'\in\Lambda}\sum_{\lambda\in\Lambda}|a(\lambda,\lambda')|),$$

we have that

$$\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{C}_{q,w}}\sup_{\lambda\in\Lambda}\|w^{-1}(\lambda,\cdot)\|_{\ell^{q'}}.$$

(iii) By Hölder inequality, (2.9) holds. □

In the next proposition, we show that $\mathcal{C}_{q,w}$ is a Banach algebra.

Proposition 2.3. *Let $1 \le q \le \infty$ and u be a companion weight matrix of the weight w, that is, w and u satisfy (2.4). Assume that $uw^{-1} \in \mathcal{C}_{q',u_0}$. Then for any $A, B \in \mathcal{C}_{q,w}$*

$$\|AB\|_{\mathcal{C}_{q,w}} \le C_1\|A\|_{\mathcal{C}_{q,w}}\|B\|_{\mathcal{C}_{q,w}}, \tag{2.10}$$

where $C_1 = 2^{d/q+1}\alpha\|uw^{-1}\|_{\mathcal{C}_{q',u_0}}$.

Proof. Let $A, B \in \mathcal{C}_{q,w}$. From the definition of the companion weight matrix, we have that

$$\begin{aligned}\|AB\|_{\mathcal{C}_{q,w}} &= \alpha\Big(\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\big(\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)b(\tilde\lambda,\lambda')|w(\lambda,\lambda')\big)^q\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)^{1/q}\\ &\le \alpha\Big(\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\big(\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)w(\lambda,\tilde\lambda)b(\tilde\lambda,\lambda')|u(\tilde\lambda,\lambda')\big)^q\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)^{1/q}\\ &\quad+\alpha\Big(\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\big(\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)u(\lambda,\tilde\lambda)b(\tilde\lambda,\lambda')|w(\tilde\lambda,\lambda')\big)^q\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)^{1/q}\\ &=: J_1+J_2 \end{aligned}\tag{2.11}$$

For any $\lambda', \tilde\lambda \in \Lambda$ there exists $\ell \in \mathbb{Z}^d$ such that $\tilde\lambda - \lambda' \in \ell + [0,1)^d$, so

$$\begin{aligned} J_1^q &\le \alpha^q\Big(\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\sum_{\ell\in\mathbb{Z}^d}\sum_{\tilde\lambda\in\Lambda}\big(|a(\lambda,\tilde\lambda)|w(\lambda,\tilde\lambda)\chi_{k-\ell+(-1,1)^d}(\lambda-\tilde\lambda)\\ &\qquad |b(\tilde\lambda,\lambda')|u(\tilde\lambda,\lambda')\chi_{\ell+[0,1)^d}(\tilde\lambda-\lambda')\big)^q\Big)\\ &\le \alpha^q\Big(\sum_{k\in\mathbb{Z}^d}\alpha\sum_{\ell\in\mathbb{Z}^d}\big(\sup_{\lambda,\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|w(\lambda,\tilde\lambda)\chi_{k-\ell+(-1,1)^d}(\lambda-\tilde\lambda)\big)^q\\ &\qquad\times\big(\sup_{\tilde\lambda,\lambda'\in\Lambda}|b(\tilde\lambda,\lambda')|u(\tilde\lambda,\lambda')\chi_{\ell+[0,1)^d}(\tilde\lambda-\lambda')\big)\\ &\qquad\times\big(\alpha\sum_{\ell\in\mathbb{Z}^d}\sup_{\tilde\lambda,\lambda'\in\Lambda}|b(\tilde\lambda,\lambda')|u(\tilde\lambda,\lambda')\chi_{\ell+[0,1)^d}(\tilde\lambda-\lambda')\big)^{q-1}\Big)\\ &\le 2^d\alpha^{2q}\|A\|^q_{\mathcal{C}_{q,w}}\|B\|^q_{\mathcal{C}_{1,u}} \end{aligned}\tag{2.12}$$

and hence

$$J_1 \le 2^{d/q}\alpha\|uw^{-1}\|_{\mathcal{C}_{q',u_0}}\|A\|_{\mathcal{C}_{q,w}}\|B\|_{\mathcal{C}_{q,w}}.\tag{2.13}$$

By the same computation, we have that

$$\|AB\|_{\mathcal{C}_{q,w}} \le 2^{d/q+1}\alpha\|uw^{-1}\|_{\mathcal{C}_{q',u_0}}\|A\|_{\mathcal{C}_{q,w}}\|B\|_{\mathcal{C}_{q,w}}.\tag{2.14}$$

□

For $\tau > 0$, we define the closed ball B_τ on $\Lambda\times\Lambda$ by

$$B_\tau = \{(\lambda,\lambda')\in\Lambda\times\Lambda : |\lambda-\lambda'|\le\tau\}\tag{2.15}$$

and denote by χ_{B_τ} the characteristic function on B_τ.

To establish the inverse-closedness of $\mathcal{C}_{q,w}$ we prove the following paracompact estimate.

Proposition 2.4. *Let $1 \le q \le \infty$, w be a weight matrix on $\Lambda\times\Lambda$ and u be the companion weight matrix of w. We assume that $uw^{-1} \in \mathcal{C}_{q',u_0}$ and there exist a positive constant C_2 and $0 < \theta < 1$ such that*

$$\inf_{\tau\ge 0}\{\Delta_\tau + t\Omega_\tau\} \le C_2 t^\theta \qquad \textit{for any } t \ge 1,\tag{2.16}$$

where

$$\Delta_\tau = \sum_{|k|\le\tau+1} \sup_{|\lambda-\lambda'|\le\tau} u(\lambda,\lambda')\chi_{k+[0,1)^d}(\lambda-\lambda')$$

and

$$\Omega_\tau = \|uw^{-1}\chi_{B_\tau{}^c}\|_{\mathcal{C}_{q',u_0}}.$$

Then for any $A, B \in \mathcal{B}_{q,w}$ *with* $\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{C}_{q,w}}$ *and* $\|B\|_{\mathcal{B}(\ell^2)} \le \|B\|_{\mathcal{C}_{q,w}}$*,*

$$\|AB\|_{\mathcal{C}_{q,w}} \le C_2\alpha^2 2^{d/q+1}\|A\|_{\mathcal{C}_{q,w}}\|B\|_{\mathcal{C}_{q,w}}\bigg(\Big(\frac{\|A\|_{\mathcal{B}(\ell^2)}}{\|A\|_{\mathcal{C}_{q,w}}}\Big)^{1-\theta} + \Big(\frac{\|B\|_{\mathcal{B}(\ell^2)}}{\|B\|_{\mathcal{C}_{q,w}}}\Big)^{1-\theta}\bigg). \tag{2.17}$$

Proof. Let $A = \big(a(\lambda,\lambda')\big)_{\lambda,\lambda'\in\Lambda} \in \mathcal{C}_{q,w}$ and $B = \big(b(\lambda,\lambda')\big)_{\lambda,\lambda'\in\Lambda} \in \mathcal{C}_{q,w}$ with $\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{C}_{q,w}}$ and $\|B\|_{\mathcal{B}(\ell^2)} \le \|B\|_{\mathcal{C}_{q,w}}$. Since u is a companion weight matrix of w, we have that

$$\begin{aligned}
\|AB\|_{\mathcal{C}_{q,w}} &= \alpha\Big(\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\big(\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)b(\tilde\lambda,\lambda')|w(\lambda,\lambda')\big)^q\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)^{1/q}\\
&\le \alpha\Big(\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\big(\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|w(\lambda,\tilde\lambda)|b(\tilde\lambda,\lambda')|u(\tilde\lambda,\lambda')\big)^q\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)^{1/q}\\
&\quad + \alpha\Big(\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\big(\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|u(\lambda,\tilde\lambda)|b(\tilde\lambda,\lambda')|w(\tilde\lambda,\lambda')\big)^q\chi_{k+[0,1)^d}(\lambda-\lambda')\Big)^{1/q}\\
&=: K_1 + K_2.
\end{aligned} \tag{2.18}$$

Since for any $\lambda, \lambda' \in \Lambda$, $|b(\lambda,\lambda')| \le \|B\|_{\mathcal{B}(\ell^2)}$, we have that

$$\begin{aligned}
K_1 &\le \alpha\|B\|_{\mathcal{B}(\ell^2)}\Big(\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\big(\sum_{\tilde\lambda\in\Lambda}\big(|a(\lambda,\tilde\lambda)|w(\lambda,\tilde\lambda)u(\tilde\lambda,\lambda')\\
&\qquad \chi_{k+[0,1)^d}(\lambda-\lambda')\chi_{B_\tau}(\tilde\lambda-\lambda')\big)^q\Big)^{1/q}\\
&\qquad + \alpha\Big(\sum_{k\in\mathbb{Z}^d}\sup_{\lambda,\lambda'\in\Lambda}\big(\sum_{\tilde\lambda\in\Lambda}|a(\lambda,\tilde\lambda)|w(\lambda,\tilde\lambda)|b(\tilde\lambda,\lambda')|u(\tilde\lambda,\lambda')\\
&\qquad \chi_{k+[0,1)^d}(\lambda-\lambda')\chi_{B_\tau{}^c}(\tilde\lambda-\lambda')\big)^q\Big)^{1/q}\\
&\qquad =: L_1 + L_2
\end{aligned} \tag{2.19}$$

Observing that for $\lambda, \lambda', \tilde{\lambda} \in \Lambda$ there exists $\ell \in \mathbb{Z}^d$ such that $\tilde{\lambda} - \lambda' \in \ell + [0,1)^d$ and if $\lambda - \lambda' \in k + [0,1)^d$, then $\lambda - \tilde{\lambda} \in k - \ell + (-1,1)^d$, so we have that

$$\begin{aligned}
L_1^q &\le \alpha^q \|B\|_{\mathcal{B}(\ell^2)}^q \Big(\sum_{k\in\mathbb{Z}^d} \sup_{\lambda,\lambda'\in\Lambda} \sum_{\ell\in\mathbb{Z}^d} \sum_{\tilde{\lambda}\in\Lambda} \big(|a(\lambda,\tilde{\lambda})| w(\lambda,\tilde{\lambda})\big)^q u(\tilde{\lambda},\lambda') \chi_{k-\ell+(-1,1)^d}(\lambda-\tilde{\lambda}) \\
&\quad \times \chi_{(\ell+[0,1)^d)\cap B_\tau}(\tilde{\lambda}-\lambda') \big(\sum_{\ell\in\mathbb{Z}^d} \sum_{\tilde{\lambda}\in\Lambda} u(\tilde{\lambda},\lambda') \chi_{(\ell+[0,1)^d)\cap B_\tau}(\tilde{\lambda}-\lambda')\big)^{q-1} \Big) \\
&\le \alpha^q \|B\|_{\mathcal{B}(\ell^2)}^q \Big(\sum_{k\in\mathbb{Z}^d} \alpha \sum_{\ell\in\mathbb{Z}^d} \big(\sup_{\lambda,\tilde{\lambda}\in\Lambda} |a(\lambda,\tilde{\lambda})| w(\lambda,\tilde{\lambda}))^q \chi_{k-\ell+(-1,1)^d}(\lambda-\tilde{\lambda})\big) \\
&\quad \times \big(\sup_{\tilde{\lambda},\lambda'\in\Lambda} u(\tilde{\lambda},\lambda') \chi_{(\ell+[0,1)^d)\cap B_\tau}(\tilde{\lambda}-\lambda')\big) \\
&\quad \times \alpha^{q-1} \big(\sum_{\ell\in\mathbb{Z}^d} \sup_{\tilde{\lambda},\lambda'\in\Lambda} u(\tilde{\lambda},\lambda') \chi_{(\ell+[0,1)^d)\cap B_\tau}(\tilde{\lambda}-\lambda')\big)^{q-1} \Big) \\
&\le \alpha^{2q} 2^d \|A\|_{\mathcal{C}_{q,w}}^q \|B\|_{\mathcal{B}(\ell^2)}^q \Delta_\tau^q
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
L_2^q &\le \alpha^q \Big(\sum_{k\in\mathbb{Z}^d} \sup_{\lambda,\lambda'\in\Lambda} \Big(\sum_{\ell\in\mathbb{Z}^d} \sum_{\tilde{\lambda}\in\Lambda} \big(|a(\lambda,\tilde{\lambda})| w(\lambda,\tilde{\lambda}))^q \chi_{k-\ell+(-1,1)^d}(\lambda-\tilde{\lambda})\big) \\
&\quad \times \big(|b(\tilde{\lambda},\lambda')| u(\tilde{\lambda},\lambda') \chi_{(\ell+[0,1)^d)\cap B_\tau{}^c}(\tilde{\lambda}-\lambda')\big)\Big) \\
&\quad \times \Big(\sum_{\ell\in\mathbb{Z}^d} \sum_{\tilde{\lambda}\in\Lambda} |b(\tilde{\lambda},\lambda')| u(\tilde{\lambda},\lambda') \chi_{(\ell+[0,1)^d)\cap B_\tau{}^c}(\tilde{\lambda}-\lambda')\Big)^{q-1}\Big) \\
&\le \alpha^q \Big(\sum_{k\in\mathbb{Z}^d} \alpha \sum_{\ell\in\mathbb{Z}^d} \big(\sup_{\lambda,\tilde{\lambda}\in\Lambda} (|a(\lambda,\tilde{\lambda})| w(\lambda,\tilde{\lambda}))^q \chi_{k-\ell+(-1,1)^d}(\lambda-\tilde{\lambda})\big) \\
&\quad \times \big(\sup_{\tilde{\lambda},\lambda\in\Lambda} |b(\tilde{\lambda},\lambda')| u(\tilde{\lambda},\lambda') \chi_{(\ell+[0,1)^d)\cap B_\tau{}^c}(\tilde{\lambda}-\lambda')\big) \\
&\quad \times \alpha^{q-1} \Big(\sum_{\ell\in\mathbb{Z}^d} \sup_{\tilde{\lambda},\lambda'\in\Lambda} |b(\tilde{\lambda},\lambda')| u(\tilde{\lambda},\lambda') \chi_{(\ell+[0,1)^d)\cap B_\tau{}^c}(\tilde{\lambda}-\lambda')\big)^{q-1}\Big) \\
&= \Omega_\tau^q \alpha^{2q} 2^d \|A\|_{\mathcal{C}_{q,w}}^q \|B\|_{\mathcal{C}_{q,w}}^q
\end{aligned} \tag{2.21}$$

Since $\|B\|_{\mathcal{B}(\ell^2)} \le \|B\|_{\mathcal{C}_{q,w}}$, it follows from (2.16), (2.19), (2.20) and (2.21) that

$$\begin{aligned}
K_1 &\le \alpha^2 2^{d/q} \|A\|_{\mathcal{C}_{q,w}} \|B\|_{\mathcal{B}(\ell^2)} \Big(\Delta_\tau + \Omega_\tau \frac{\|B\|_{\mathcal{C}_{q,w}}}{\|B\|_{\mathcal{B}(\ell^2)}}\Big) \\
&\le C_2 \alpha^2 2^{d/q} \|A\|_{\mathcal{C}_{q,w}} \|B\|_{\mathcal{B}(\ell^2)} \Big(\frac{\|B\|_{\mathcal{C}_{q,w}}}{\|B\|_{\mathcal{B}(\ell^2)}}\Big)^\theta
\end{aligned} \tag{2.22}$$

By the same computation we have that

$$K_2 \le C_2 \alpha^2 2^{d/q} \|A\|_{\mathcal{B}(\ell^2)} \|B\|_{\mathcal{C}_{q,w}} \Big(\frac{\|A\|_{\mathcal{C}_{q,w}}}{\|A\|_{\mathcal{B}(\ell^2)}}\Big)^\theta. \tag{2.23}$$

Combining (2.22) and (2.23), (2.17) holds.

□

In (2.17), we put $A = B$ to get

$$\|A^2\|_{\mathcal{C}_{q,w}} \le C_3 \|A\|_{\mathcal{C}_{q,w}}^{1+\theta} \|A\|_{\mathcal{B}(\ell^2)}^{1-\theta}, \tag{2.24}$$

where $C_3 = C_2 \alpha^2 2^{d/q+2}$.

Proposition 2.5. *Under the assumptions of Proposition 2.4 for a positive integer n and $A \in \mathcal{C}_{q,w}$, we have that*

$$\|A^n\|_{\mathcal{C}_{q,w}} \le (C_1 C_3)^{\log_2 n} \Big(\frac{\|A\|_{\mathcal{C}_{q,w}}}{\|A\|_{\mathcal{B}(\ell^2)}} \Big)^{\frac{(1+\theta)}{\theta} n^{\log_2 (1+\theta)}} \|A\|_{\mathcal{B}(\ell^2)}^n. \tag{2.25}$$

Proof. Let $A \in \mathcal{C}_{q,w}$ and n be a positive integer. We write $n = \sum_{j=0}^N \varepsilon_j 2^j$, where $\varepsilon_N = 1$ and $\varepsilon_j \in \{0, 1\}$. We put

$$n_\ell = \varepsilon_\ell + 2n_{\ell+1}, \quad n_N = \varepsilon_N, \qquad \text{for } \ell = 0, \cdots, N-1.$$

We assume that $\|A\|_{\mathcal{B}(\ell^2)} = 1$. Since $N \le \log_2 n$, it follows from (2.10), (2.24) that

$$\sum_{k=0}^N \varepsilon_k (1+\theta)^k \le \frac{1+\theta}{\theta}(1+\theta)^N,$$

and

$$\begin{aligned}
\|A^n\|_{\mathcal{C}_{q,w}} &\le C_1 \|A\|_{\mathcal{C}_{q,w}}^{\varepsilon_0} \|A^{2n_1}\|_{\mathcal{C}_{q,w}} \\
&\le C_1 C_3 \|A\|_{\mathcal{C}_{q,w}}^{\varepsilon_0} \|A^{n_1}\|_{\mathcal{C}_{q,w}}^{1+\theta} \\
&\le C_1^2 C_3 \|A\|_{\mathcal{C}_{q,w}}^{\varepsilon_0+\varepsilon_1(1+\theta)} \|A^{2n_2}\|_{\mathcal{C}_{q,w}}^{1+\theta} \\
&\le C_1^2 C_3^2 \|A\|_{\mathcal{C}_{q,w}}^{\varepsilon_0+\varepsilon_1(1+\theta)} \|A^{n_2}\|_{\mathcal{C}_{q,w}}^{(1+\theta)^2} \\
&\cdots \\
&\le C_1^N C_3^N \|A\|_{\mathcal{C}_{q,w}}^{(\varepsilon_0+\varepsilon_1(1+\theta)+\cdots+\varepsilon_N(1+\theta)^N)} \\
&\le (C_1 C_3)^{\log_2 n} \|A\|_{\mathcal{C}_{q,w}}^{\frac{1+\theta}{\theta} \cdot n^{\log_2(1+\theta)}}.
\end{aligned} \tag{2.26}$$

If $\|A\|_{\mathcal{B}(\ell^2)} \ne 1$, then replacing A in (2.26) by $A/\|A\|_{\mathcal{B}(\ell^2)}$ we have the relation (2.25). □

Now we can prove the inverse-closedness of the Banach algebra $\mathcal{C}_{q,w}$ in $\mathcal{B}(\ell^2)$.

Theorem 2.6. *Let $1 \le q \le \infty$ and w be a weight matrix on $\Lambda \times \Lambda$. Under the assumptions of Proposition 2.4, the Banach algebra $\mathcal{C}_{q,w}$ is inverse-closed in $\mathcal{B}(\ell^2)$, that is, if $A \in \mathcal{C}_{q,w}$ and $A^{-1} \in \mathcal{B}(\ell^2)$, then $A^{-1} \in \mathcal{C}_{q,w}$.*

Proof. Let $A \in \mathcal{C}_{q,w}$ and $A^{-1} \in \mathcal{B}(\ell^2)$. By (2.25), we have the following estimate of $\|A^n\|_{\mathcal{C}_{q,w}}$ that for a positive integer n and $A \in \mathcal{C}_{q,w}$ with $\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{C}_{q,w}}$,

$$\|A^n\|_{\mathcal{C}_{q,w}} \le (C_1 C_3)^{\log_2 n} \Big(\frac{\|A\|_{\mathcal{C}_{q,w}}}{\|A\|_{\mathcal{B}(\ell^2)}} \Big)^{\cdot \frac{(1+\theta)}{\theta} n^{\log_2 (1+\theta)}} \|A\|_{\mathcal{B}(\ell^2)}^n. \tag{2.27}$$

We denote by A^* the conjugate transpose of A. The product A^*A is a positive operator on ℓ^2 and by the invertibility of A we have that

$$\tilde{C}_1 I \le A^* A \le \tilde{C}_2 I, \tag{2.28}$$

where $\tilde{C}_1 = \|A^{-1}\|^{-2}_{\mathcal{B}(\ell^2)}$ and $\tilde{C}_2 = \|A\|^2_{\mathcal{B}(\ell^2)}$. We set

$$B = I - \frac{A^*A}{\tilde{C}_1 + \tilde{C}_2}.$$

Then

$$\frac{\tilde{C}_1}{\tilde{C}_1 + \tilde{C}_2} \leq \|B\|_{\mathcal{B}(\ell^2)} \leq \frac{\tilde{C}_2}{\tilde{C}_1 + \tilde{C}_2} = r_0 < 1.$$

Suppose that $\|B\|_{\mathcal{C}_{q,w}} \leq \|B\|_{\mathcal{B}(\ell^2)}$. Then

$$\sum_{n=0}^{\infty} \|B^n\|_{\mathcal{C}_{q,w}} \leq \sum_{n=1}^{\infty} r_0^n < \infty. \tag{2.29}$$

Assume that $\|B\|_{\mathcal{C}_{q,w}} > \|B\|_{\mathcal{B}(\ell^2)}$. Since from (2.27) $\limsup_{n\to\infty} \|B^n\|^{1/n}_{\mathcal{C}_{q,w}} \leq r_0 < 1$, we have that $\sum_{n=1}^{\infty} \|B^n\|_{\mathcal{C}_{q,w}} < \infty$. Observing that $A^{-1} = (A^*A)^{-1}A^* = (\tilde{C}_1 + \tilde{C}_2)^{-1}(I - B)^{-1}A^*$, we have that

$$\|A^{-1}\|_{\mathcal{C}_{q,w}} \leq (\tilde{C}_1 + \tilde{C}_2)^{-1}(\|I\|_{\mathcal{C}_{q,w}} + \sum_{n=1}^{\infty} \|B^n\|_{\mathcal{C}_{q,w}})\|A^*\|_{\mathcal{C}_{q,w}} < \infty. \tag{2.30}$$

Hence $A^{-1} \in \mathcal{C}_{q,w}$. □

In the next remark, we show that a polynomial weight and subexponential weight satisfy the condition (2.4) and (2.16).

Remark 2.7. *We consider the case $\Lambda = \mathbb{Z}^d$, and polynomial weight matrices and subexponential weight matrices.*

Let $1 \leq q \leq \infty$ and $1/q' = 1 - 1/q$. For $\alpha > d/q'$, consider the polynomial weight matrix $w_\alpha(i,j) = (1 + |i-j|)^\alpha$. The constant weight $u_\alpha(i,j) = 2^\alpha$ for any $i,j \in \mathbb{Z}^d$ satisfies the companion weight condition (2.4). For $\tau \geq 0$,

$$\Delta_\tau = 2^\alpha(2\tau + 3)$$

and

$$\Omega_\tau \leq 4^\alpha(\alpha q' - d)^{-1/q'}(\tau + 1)^{-\alpha + d/q'}.$$

We have that for $t \geq 1$,

$$\begin{aligned}
&\inf_{\tau \geq 0}\{\Delta_\tau + t\Omega_\tau\} \\
&\quad \leq \inf_{\tau \geq 0}\Big\{\sum_{|k| \leq \tau+1} 4^\alpha\Big((2\tau+3)^d + t(\alpha q' - d)^{-1/q'}(\tau+1)^{-\alpha+d/q'}\Big)\Big\} \\
&\quad \leq \inf_{\tau \geq 0} 4^\alpha 3^d\big(1 + (\alpha q' - d)^{-1/q'}\big)\big((\tau+1)^d + t(\tau+1)^{-\alpha+d/q'}\big) \\
&\quad \leq 2^{2\alpha+1}3^d\big(1 + (\alpha q' - d)^{-1/q'}\big)t^{\frac{d}{d/q+\alpha}},
\end{aligned}$$

where in the last inequality τ satisfies the equation $(\tau+1)^d = t(\tau+1)^{-\alpha+d/q'}$. Hence the polynomial weight matrix satisfies (2.16).

Next, for $D > 0$ and $0 < \delta < 1$, we consider the subexponential weight matrix $w(i,j) = e^{D|i-j|^\delta}$. The weight matrix $u(i,j) = e^{D(2^\delta - 1)|i-j|^\delta}$ satisfies the companion weight matrix condition (2.4). Following Remark 3.4 of [13]*, we can see that w satisfies (2.16).*

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